Condensed Thoughts

Based on

* ideas learned from Collaboration colleagues
  (Cordova, Dumitrescu, Thorngren)
* more mathematical reframing by Gaiotto & Johnson-Freyd
* detailed discussions (project?) with Freed & Hopkins
* older ideas on finite homotopy theories w/ Freed, Hopkins, Lurie
  ~~partially~~ made precise by Scheimbauer-Walde
* established ideas in (3D) physics/cond matter literature (Kitaev)

Theme: Calculus of topological defects in QFT

Emphasis: those coming from FHTs. (finite htpy theories)

Specific Goal: "Explain" Condensation of defects

or, What do we need in the target category to execute that?

"Deliverables": we isolated two ingredients for ↑

* Dirichlet boundary condition (or, end of a defect)
* Ostrik's principle : correspondence between

$\left\{\begin{array}{l}\text{modules over algebras}\\ \text{in higher categories}\end{array}\right\} \longleftrightarrow \{\text{Internal algebra objects}\}$

Applications [Recovering a pointed space from its symmetry TFT]

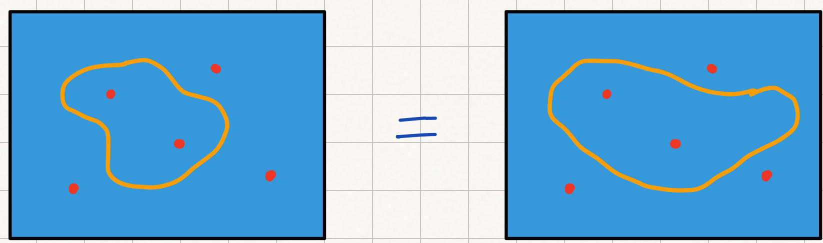
Recovering a (π-finite) space from its quantization ONLY POSSIBLE UP TO EM DUALITY
↳ (what is a quantum homotopy E_n type?)

Moral: Filtration of defects by condensatiousness
$\longleftrightarrow$ Postnikov filtration of the space

(CONDENSATION $\updownarrow$ CONNECTIVITY)

Recap 1. Topological defects (extended operators)

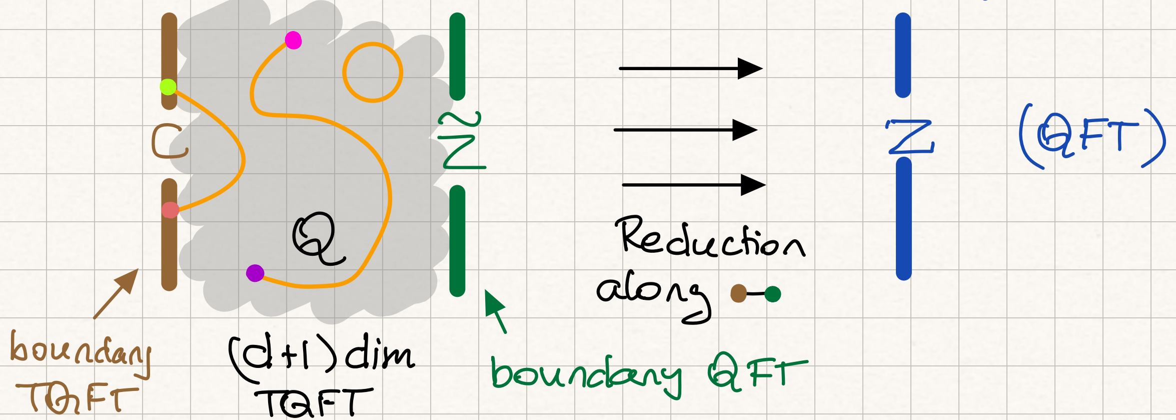
* can move freely in the vacuum



topological operator

QFT operator

* Our operators will be implemented by a quiche (Q, C)
(crust)



Interpretation: Action of $\mathrm{End}_Q(C)$ on Z.

Mostly I'll be concerned with defects in the crust C.

Recap 2. Calculus of defects

* Internal to Q:

linking sphere S^ℓ (bundle over D)

support of defect: codim $(\ell+1)$ in $(d+1)$

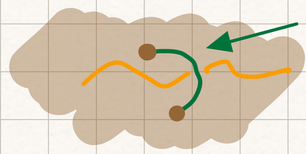
$Q(S^\ell)$: defines a TQFT of dim $(d+1-\ell)$

(Quantum) defect local label: boundary theory for $Q(S^\ell)$

$= $ object in $\mathrm{Hom}_{\Omega^\ell \mathcal{T}}(\mathbb{1}^\ell, Q(S^\ell))$

Explanation: Remove a thickened D, place a locally (on D) built state on the spherical boundary

Such states are built from boundary theories.

* On the crust  linking disk $(D^e, \partial D^e)$

$Q(D^e, \partial D^e)$: TQFT of dim. $(d+1-e)$ ... same story on labels

Example $\mathcal{T}$: $(d+1)$-category of d-algebras $Alg(Alg(Alg(\dots)..)$

If Q is built from a π-finite space X

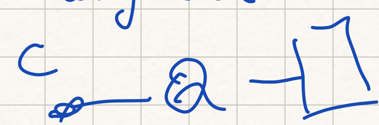
C is the canonical Dirichlet condition

= one base-point in each component

$$Q(S^e) \text{ "="} Q_{d-e}(Map(S^e; X))$$

= a $(d-e)$-algebra built by iterated crossed products with the $\pi_i\, Map(S^e; X)$ in descending order

$$Q(D^e, \partial D^e) \text{ "="} \bigoplus_{\pi_0 X} Q_{d-e}(\Omega^e X)$$



Boundary theories for these are modules in $(d-e-1)$ algebras

$\equiv$ $(d-e)$ dim. TQFTs with actions of $\Omega Map(S^e; X), \dots$

Some can be obtained from spaces $Y \longrightarrow Map(S^e; X)$ by quantizing the homotopy fiber

(The loop space of the base acts on the homotopy fiber)

Addenda: X may carry a generalized Dijkgraaf-Witten twist.

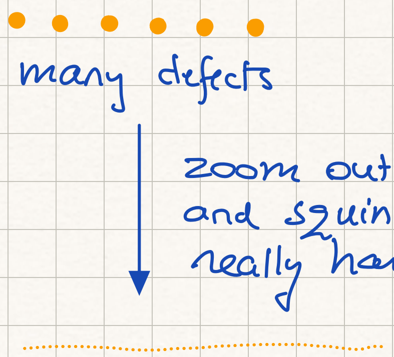
Caution (orientations)

The spheres rotate along the defect.

the $so(e+1)$ action on $Map(S^e; X)$ may be nontrivial

$\Rightarrow$ care needed for tangential structures

Physicist's wish list for a defect • of codimension ℓ



many defects

zoom out and squint really hard

defect of codimension $(\ell-1)$.

More seriously:

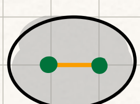
* Condensed defects should be endable, 
* a closed condensed defect can be ripped open and closed back up
* This allows any linking defect to slide away.
* Iterating: The support of k-fold condensed defects may be shrunk by k dimensions.

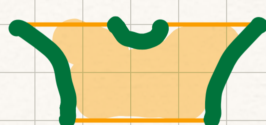
This phenomenon accounts for claims of a "calculus of defects" that mixes dimensions.

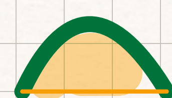
(Some of the defects in the product may be shrunk).

Interpretation [Loose with framings and duals!]

Endable defects of Codim ℓ $\longrightarrow$ algebra objects in defects codim $(\ell+1)$

 algebra in one lower dimension

 m

 u

But, m need not be an isomorphism so need a condition on the end to recover  from 

(1) (2)

This is the (strong) Dirichlet condition

Example At the vector space level (line, end) we need a collection of ends so that the (2) contain (1) in their span.

Dirichet condition on a boundary theory

$(d+1)$-dim. TQFT $\mathcal{Q}$ (e.g. $\mathcal{Q}_d(X)$) $\longleftrightarrow$ $\mathcal{Q}(pt) \in \mathcal{J}$, $(d+1)$ cat.

Boundary theory $\mathcal{C}$ $\longleftrightarrow$ $\beta := \mathcal{C}(pt) \in \mathrm{Hom}_{\mathcal{J}}(\mathbb{1}, \mathcal{Q}(pt))$

$\mathcal{C}$ is Dirichlet iff:

(i) The $\mathrm{End}_{\mathcal{J}}(\mathbb{1})$-module categories $\mathrm{Hom}_{\mathcal{J}}(\mathbb{1}, \mathcal{Q}(pt))$ and $\mathrm{Mod}\text{-}\beta^R \circ \beta$ are equivalent (Ostrik correspondence)

("β generates all boundary theories for $\mathcal{Q}$")

(ii) $\mathcal{Q}(pt)$ is local at $\mathbb{1}$: For $\forall\, Y \in \mathrm{Ob}\,\mathcal{J}$, have an equivalence

$$\mathrm{Hom}(\mathcal{Q}(pt), Y) \longrightarrow \mathrm{Hom}_{\mathrm{End}(\mathbb{1})}\big(\mathrm{Hom}(\mathbb{1}, \mathcal{Q}(pt));\ \mathrm{Hom}(\mathbb{1}, Y)\big)$$

("$\mathcal{Q}$ is equivalent to the TQFT generated by $\beta^R \circ \beta$")

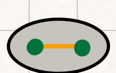
Example: $\mathcal{J}$ = 2-category of (Alg, Bimod, Intertwiners)

Y–$\mathcal{Q}$ bimodules $\longleftrightarrow$ (cocontinuous) functors $\mathcal{Q}\text{-Mod} \longrightarrow Y\text{-Mod}$.

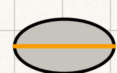
Remark: It should be the case that (i) and (ii) can be reworded in the language of retracts as in Gaiotto & Johnson-Freyd as $\mathrm{Id}_{\mathcal{Q}}$ is a retract of $\beta \circ \beta^L \in \mathrm{End}(\mathcal{Q})$.

Dirichlet condition on the end of a defect

Treat the defect as an embedded TQFT and repeat.



condenser, algebra defect   codim $(\ell+1)$



condensate, codim ℓ

Quasi-theorem* A defect with a Dirichet end may be cut open and closed up again.

*provable after making more precise

Examples of Dirichlet conditions

* FHT: $Y \to X$ is a Dirichlet condition for dimension $(d+1)$

$$\Longleftrightarrow \begin{cases} \pi_0 Y \to \pi_0 X & \text{onto} \\ \pi_d Y \to \pi_d X & \text{zero}^* \end{cases} \Big\updownarrow \text{ EM dual.}$$

(*exceptions if $d=2$ & $\pi_1 X$ nonab.)

eg: $d=1$, $X = BG$ $Y = BH$

Dirichlet ∂: a representation containing every irrep of G.

If G abelian, only happens for Ind_H^G if $H \to G$ is trivial.

* In particular, if X is connected, $(pt) \to X$ is Dirichlet.

(canonical Dirichlet condition:

the homotopy fiber is ΩX with its translation action).

* Dirichlet ends for FHT defects:

$Y \longrightarrow \Omega^\ell X$ must satisfy the $\pi_0, \pi_{d-\ell}$ as above

* Fusion categories Φ

Φ indecomposable $\Longrightarrow$ any nonzero module cat M is Dirichlet

$$\Phi \overset{ME}{\equiv} \mathrm{End}_\Phi(M).$$

$\Phi \cong \Phi_1 \oplus \Phi_2$, $M = \Phi_1 \oplus 0$ — not Dirichlet.

In particular, for gauge theory in dim ≥ 3

the Neumann boundary theory is Dirichlet.

Example: Condensation in 3D theories

Theorem In a simple Turaev-Viro or Reshetikhin-Turaev theory:

(1) only the transparent line operator is condensed

(2) all surface operators are condensed.

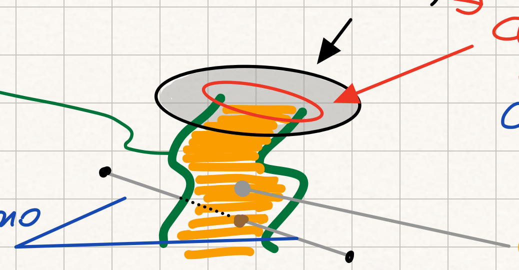
Pf (1) Only scalar point operators (simple; connected)

(2)

$$
\begin{array}{c}
\text{Surface operators} \\
\updownarrow \\
F\text{-}F \text{ bimodule cats.} \\
\updownarrow \;\; \text{(connectedness)} \\
Z(F)\text{-module cats} \\
\updownarrow \;\; \text{Ostrik} \\
\text{algebra objects in } Z(F) \text{ up to Morita equivalence}
\end{array}
$$

fixed by the end

$Z(F) \ni a$ condenser: algebra object



end of Condensate

$F \boxtimes F^{mo}$

Condensate: F-F bimodule ${}_F M_F$

bimodule functor ${}_F F_F \xrightarrow{\text{end}} {}_F M_F$ ⇝ $Z(F)$-module functor

$$
\begin{array}{c}
\text{end}' : Z(F) \longrightarrow F \boxtimes_{F\boxtimes F^{mo}} M \\
\updownarrow \\
\text{object in module category}
\end{array}
$$

Remark In RT theories, interfaces are "$Z(F)$"-modules by folding.

* Key (Non) Fact

The Dirichlet property is NOT preserved by dimensional reduction.

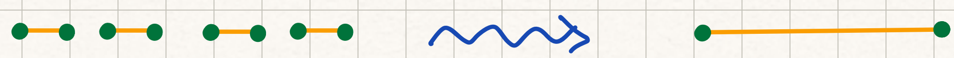
Eg : * 3D gauge theory reduced on S^1:
No ∂ condition generates the Drinfeld center upon reduction.

* $\mathrm{Maps}(M;*) \longrightarrow \mathrm{Map}(M;X)$ is NOT Dirichlet

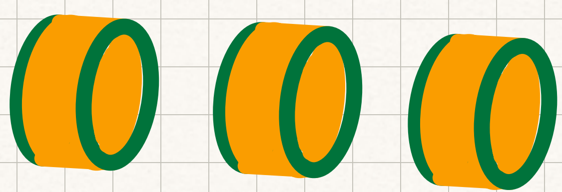
usually not connected!

Key problem : We can condense from dim. $(d-\ell)$ to $(d-\ell+1)$

But then we seem stuck because our 'end' is no longer Dirichlet :



But

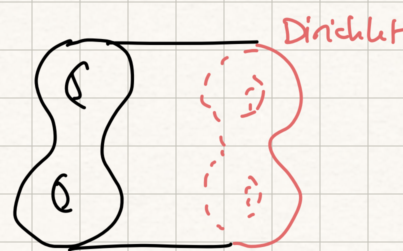


seems not condensable ?

Theorem (FHT case but probably in general)

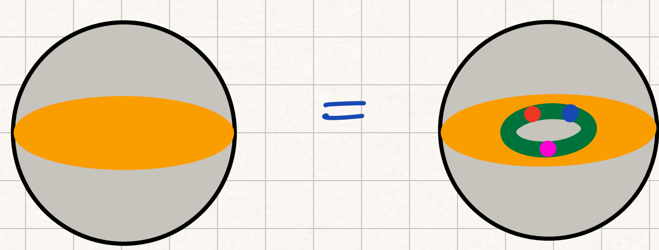
The dimensional reduction of a Dirichlet end can be made Dirichlet after embedding additional self defects.

Example We can span the space of states for a d-manifold by inserting higher codim defects in the Dirichlet boundary.



Theorem At the top dimension (if $d+1 \geq 3$) we can 'rip open' the defect without changing correlators by embedding suitable self-defects in the Dirichlet end.

defect



same vectors in the space for the (defective) sphere

$(S^2;$ equator$)$

Classical Condensation: Crust defects in the $(d+1)$ dim Quiche from a space X with its Dirichlet (basepoint) condition

$Q_m(Y) :=$ quantization as algebra in an m-category

Codimension of defects	Local quantum labels are modules over...	Contributing homotopy groups, placed in dim $[k]$: all/ $\boxed{\text{new}}$ defects	Condensed
d (pts)	$Q_0(\Omega^d X)$	$\boxed{\pi_d[0]}$	None
$d-1$ (lines)	$Q_1(\Omega^{d-1} X)$	$\boxed{\pi_{d-1}[0]}, \pi_d[1]$	$\pi_d[1]$
$d-2$ (surfaces)	$Q_2(\Omega^{d-2} X)$	$\boxed{\pi_{d-2}[0]}, \pi_{d-1}[1], \pi_d[2]$	$\pi_{d-1}[1], \pi_d[2]$
k	$Q_{d-k}(\Omega^k X)$	$\boxed{\pi_k[0]}, \pi_{k+1}[1], \pi_{k+2}[2], \dots$	$\pi_{k+1}[1], \pi_{k+2}[2], \dots$
2	$Q_{d-2}(\Omega X)$	$\boxed{\pi_2[0]}, \pi_3[1] \dots, \pi_d[d-2]$	$\pi_3[1] \dots, \pi_d[d-2]$
1 (interface)	$Q_{d-1}(X)$	$\boxed{\pi_1[0]}, \pi_2[1], \dots \pi_d[d-1]$	$\pi_2[1], \pi_3[2] \dots, \pi_d[d-1]$

"Classically" condensed defects are those in positive htpy degree. High Postnikov fibers are increasingly condensed.

Classical shrinking of support: maps from manifolds to highly connected spaces can be supported in high co-dimension.

The listing also suggests that $[Q_d(X), Q_{d+1}(\Omega X)]$ determines $(X, *)$.

That is the case, and actions of this quiche on a QFT $\Longleftrightarrow$ actions of ΩX.

Shrinking of support

Classical:

Assume Z connected, $\pi_1 Z$ abelian, acting trivially on $\pi_* Z$.

Then: $\pi_k \,\mathrm{Based\,Map}(M;Z) \cong \pi_k \,\mathrm{Map}(M,Z)$ whenever $\pi_k Z = 0$.

M manifold $\Rightarrow$ BasedMap can be retracted to codim 1.

Z highly connected $\Rightarrow$ can shrink the support of maps without damaging the low homotopy groups.

Quantum:

Even if X is highly connected, high-dimensional defects need not be condensed: because you could stack a TQFT w/o boundary conditions on top of the defect. But that's the only problem:

Theorem If the quantum defect label admits a Dirichlet ∂, and maps to the ≥ 0 part of $\Omega^k X$, then the defect is condensed and its support may be retracted to codim 1.

maybe theorem one day

~~Alt Theorem If the $\Omega^k X$ action on the label is induced from a high Postnikov cover, then in top dimension the defect is a constant TQFT with an embedded defect.~~

Repeating the bookkeeping for $\Theta_d(X)$ alone "doubles" the homotopy groups, because $\mathrm{Maps}(S^\ell;X)$ replace $\Omega^\ell X$.

For codim $(k+1)$ defects, space is $\mathrm{Map}(S^k;X)$;

$$\begin{cases} \pi_0, \dots, \pi_{d-k} & \text{from } X \\ \pi_k[0], \dots \pi_d[d-k] & \text{from } \Omega^k X \end{cases}$$

Positions p and $(d-p)$ of homotopy groups are indistinguishable

Appearance of $\pi_k[0]$ for $Q_{d-k}(\mathrm{Map}(S^k; X))$ can't be distinguished from occurence of $\pi_k^\vee$ in deg. $(d-k)$.

so π_k will occur in degrees k and $(d-k)$ on the list

Expected Theorem: The quantization $Q_d(X)$ determines X "up to electro-magnetic duality".

Morally: $(d+1)$-dim quantization "folds" homotopy d-types (with Dijkgraaf-Witten twist) via $\pi_k \longleftrightarrow \pi_{d-k}$

? Answer Homotopy types with "poly-k-invariants".

X has: $\pi_i(X)$, $k^\bullet \in \bigoplus_p C^p(X; \pi_{p-1}X)$ $\left(+\ \tau \in H^{d+1}(X; \mathbb{C}^*)\right)$

Morally: cochains on X = functions on the structured set $\bigoplus \pi_p(X)$

$\Rightarrow$ $(k^\bullet, \tau)$ is a function on $\underbrace{\bigoplus \pi_p(X)}_{"X"} \times \underbrace{\bigoplus \pi_p(X)^\vee}_{"X^\vee"}$

- Linear on $X^\vee$
- Of total degree $(d+1)$, if you shift $X^\vee$ to degree $(-d)$

EM duality switches X and $X^\vee$.

(That's why we can define it for linear $k^\bullet \Leftrightarrow \infty$ loop space X).

To extend this we need nonlinear (in $X^\vee$) k-invariants

Proposition This makes sense in rational homotopy and defines E_d structures on $C^*(X; \mathbb{Q})$.

No comparable statement for π-finite spaces

General procedure TQFT $\mathcal{Q}$, target cat. $\mathcal{T}$

Have an adjunction

$$\mathrm{Hom}_{\Omega^{\ell+1}\mathcal{T}}(\mathbb{1}^{\ell+1}, \mathcal{Q}(S^{\ell+1})) \cong \mathrm{End}_{\mathrm{Hom}(\mathbb{1}^\ell, \mathcal{Q}(S^\ell))}(\mathcal{Q}(D^{\ell+1}))$$

$$\begin{tikzcd} \mathbb{1}^\ell \arrow[r, "\mathcal{Q}(D^{\ell+1})"] & \mathcal{Q}(S^\ell) \arrow[l, bend left, "\mathcal{Q}(D^{\ell+1})^*"] \end{tikzcd}$$

$$\mathcal{Q}(D^{\ell+1})^* \circ_{\mathcal{Q}(S^\ell)} \mathcal{Q}(D^{\ell+1})$$

$$\Omega^\ell_{\mathcal{T}} := \underbrace{\mathrm{End}(\mathrm{End}(\ldots(\mathrm{End}(\mathbb{1}_0)\ldots)}_{\ell \text{ times}}$$

Oshik principle:

algebra objects in LHS = RHS $\longleftrightarrow$ modules over RHS (with a generator)

Localization Condition (from def of Dirichlet end)

Dirichlet-endable objects in $\mathrm{Hom}(\mathbb{1}^\ell, \mathcal{Q}(S^\ell))$ are determined by their "localization at the unit" $\mathcal{Q}(D^{\ell+1})$

Localization at $\mathcal{Q}(D^{\ell+1})$ is an equivalence

$$\boxed{\begin{array}{l} M \in \mathrm{Hom}_{\Omega^\ell \mathcal{T}}(\mathbb{1}^\ell, \mathcal{Q}(S^\ell)) \\ + \overset{(\text{Dirichlet})}{\text{end}} \in \mathrm{Hom}(\mathcal{Q}(D^{\ell+1}), M) \end{array}} \xrightarrow{\text{localize}} \mathrm{Hom}_{\Omega^\ell \mathcal{C}}(\mathcal{Q}(D^{\ell+1}), M)$$

$$\mathrm{End}_{\mathrm{Hom}(\mathbb{1}^\ell, \mathcal{T}(S^\ell))}(\mathcal{Q}(D^{\ell+1}))$$

$$\boxed{\begin{array}{c} \text{algebra object} \\ a \in \mathrm{Hom}(\mathbb{1}^{\ell+1}, \mathcal{Q}(S^{\ell+1})) \end{array}} \equiv \xleftarrow{\text{Oshik}} \quad + \text{ generator}$$

Dirichlet condition on end says that you can "condense back" to the original (defect, end).

Overall

$\mathcal{A}$ = algebra in codim-$(\ell+2)$ local defects $\mathrm{Hom}\left(\mathbb{1}_{\ell+1}, \mathcal{Q}_d(X)[S^{\ell+1}]\right)$
+ regular module

$\longrightarrow$ $\mathrm{End}_{\dots}[\mathcal{Q}(D^{\ell+1})]$ module (+ generator)

$\longrightarrow$ codim$(\ell+1)$ defect label in $\mathrm{Hom}(\mathbb{1}^{\ell}, \mathcal{Q}_d(X)[S^{\ell}])$
(+ defect end) (condensate)